Online appendix for "Corporate governance in the presence of active and passive delegated investment"

Auxiliary Result. Note that

$$\begin{aligned} R_L &= R_0 + e_{AL} + e_P \\ R_M &= R_0 + \frac{e_{AL}}{2} + e_P, \end{aligned}$$

which imply

$$\begin{aligned} e_{AL} &= 2\left(R_L - R_M\right) & (69) \\ e_P &= 2R_M - R_L - R_0. & (70) \end{aligned}$$

Using Proposition 1 and since $e_{AL} = \frac{f_A x_{AL}}{c_A}$ and $e_P = \frac{f_P x_P}{c_P}$, we get the following expressions for x_{AL} and x_P as functions of λ and the model parameters:

$$\begin{aligned} x_{AL} &= \frac{2c_A}{f_A\left(\lambda\right)}\left[\,\xi_A\left(\lambda\right)Z_L - \xi_P\left(\lambda\right)Z_M\right], & (71) \\ x_P &= \frac{c_P}{f_P\left(\lambda\right)}\left[\,2\xi_P\left(\lambda\right)Z_M - \xi_A\left(\lambda\right)Z_L - R_0\right], & (72) \end{aligned}$$

where

$$\begin{aligned} \xi_A\left(\lambda\right) &\equiv 1 + \frac{1-\eta}{\psi_A + \left(\lambda - 1\right)\left(1-\eta\right)} & (73) \\ \xi_P\left(\lambda\right) &\equiv 1 + \frac{1-\eta}{\psi_P + \left(\lambda - 1\right)\left(1-\eta\right)} & (74) \\ f_A\left(\lambda\right) &\equiv \frac{\eta\psi_A}{\psi_A + \lambda\left(1-\eta\right)} \\ f_P\left(\lambda\right) &\equiv \frac{\eta\psi_P}{\psi_P + \lambda\left(1-\eta\right)}. \end{aligned}$$

■

Lemma 2 *Consider any equilibrium given by Proposition 1. Then, the rate of return λ is decreasing in aggregate wealth W if $|c_P - c_A|$ is sufficiently small, or if $\psi_A \geq \psi_P$ and $c_A \leq \frac{\psi_A}{\psi_P} c_P$. Moreover, under either of these conditions, λ is strictly decreasing in W if $\lambda > 1$.*

Proof of Lemma 2. We present the proof for the quadratic cost functions, $c_i\left(e\right) = \frac{c_i}{2}e^2$. Note that in any equilibrium where W is strictly larger than the total AUM raised by funds, it has to be $\lambda = 1$, because otherwise $\lambda > 1$ and hence the fund investors that invest in the outside asset would strictly prefer to deviate and invest in a fund. Therefore, if $\lambda > 1$, then it has to be that W is equal to the total AUM. For this reason, to prove the lemma, it is sufficient to show that the total AUM strictly decreases with λ.

Consider any equilibrium given by Proposition 1. Then, the total AUM raised by funds is

$$\begin{aligned} & \frac{P_L}{2f_A} c_A \left(2\left(R_L - R_M\right)\right) + \frac{P_M}{f_P} c_P \left(2R_M - R_L - R_0\right) \qquad (75) \\ = \;& c_A \frac{\psi_A + \lambda(1-\eta)}{\eta \psi_A} \begin{pmatrix} \left(1 + \frac{1-\eta}{\psi_A + (\lambda-1)(1-\eta)}\right) Z_L \\ -\left(1 + \frac{1-\eta}{\psi_P + (\lambda-1)(1-\eta)}\right) Z_M \end{pmatrix} \frac{1-\eta}{\psi_A + (\lambda-1)(1-\eta)} Z_L \\ & + c_P \frac{\psi_P + \lambda(1-\eta)}{\eta \psi_P} \begin{pmatrix} 2\left(1 + \frac{1-\eta}{\psi_P + (\lambda-1)(1-\eta)}\right) Z_M \\ -\left(1 + \frac{1-\eta}{\psi_A + (\lambda-1)(1-\eta)}\right) Z_L - R_0 \end{pmatrix} \frac{1-\eta}{\psi_P + (\lambda-1)(1-\eta)} Z_M, \end{aligned}$$

Note that by the proof of Proposition 1, both funds raise positive AUM, and hence $x_{AL} > 0$ and $x_P > 0$. Moreover, λ has a finite upperbound by Lemma 8. Therefore, (2) implies that $e_{AL} = \frac{f_A x_{AL}}{c_A} > 0$ and $e_P = \frac{f_P x_P}{c_A} > 0$, and in turn, (7)-(10) imply that $R_L - R_M = \frac{1}{2} e_{AL} > 0$ and $2R_M - R_L - R_0 = e_P > 0$. Plugging in the expressions for R_L and R_M from Proposition 1 yields

$$0 > -\left(1 + \frac{1-\eta}{\psi_A + (\lambda-1)(1-\eta)}\right) Z_L + \left(1 + \frac{1-\eta}{\psi_P + (\lambda-1)(1-\eta)}\right) Z_M \qquad (76)$$

and

$$0 > -2\left(1 + \frac{1-\eta}{\psi_P + (\lambda-1)(1-\eta)}\right) Z_M + R_0 + \left(1 + \frac{1-\eta}{\psi_A + (\lambda-1)(1-\eta)}\right) Z_L, \qquad (77)$$

respectively. Multiplying (75) by $\frac{\eta}{1-\eta}$ and rearranging the terms, we get

$$\begin{aligned} & \frac{c_A}{\psi_A} \begin{bmatrix} \left(1 + \frac{1-\eta}{\psi_A + (\lambda-1)(1-\eta)}\right)^2 Z_L \\ -\left(1 + \frac{1-\eta}{\psi_A + (\lambda-1)(1-\eta)}\right)\left(1 + \frac{1-\eta}{\psi_P + (\lambda-1)(1-\eta)}\right) Z_M \end{bmatrix} Z_L \\ & + \frac{c_P}{\psi_P} \begin{bmatrix} 2\left(1 + \frac{1-\eta}{\psi_P + (\lambda-1)(1-\eta)}\right)^2 Z_M \\ -\left(1 + \frac{1-\eta}{\psi_A + (\lambda-1)(1-\eta)}\right)\left(1 + \frac{1-\eta}{\psi_P + (\lambda-1)(1-\eta)}\right) Z_L \\ -\left(1 + \frac{1-\eta}{\psi_P + (\lambda-1)(1-\eta)}\right) R_0 \end{bmatrix} Z_M \end{aligned}$$

Hence, (75) is strictly decreasing in λ if and only if

$$0 > \frac{c_A}{\psi_A}\left[\begin{array}{c} -2\left(1+\frac{1-\eta}{\psi_A+(\lambda-1)(1-\eta)}\right)\frac{(1-\eta)^2}{(\psi_A+(\lambda-1)(1-\eta))^2}Z_L \\ +\frac{(1-\eta)^2}{(\psi_A+(\lambda-1)(1-\eta))^2}\left(1+\frac{1-\eta}{\psi_P+(\lambda-1)(1-\eta)}\right)Z_M \\ +\left(1+\frac{1-\eta}{\psi_A+(\lambda-1)(1-\eta)}\right)\frac{(1-\eta)^2}{(\psi_P+(\lambda-1)(1-\eta))^2}Z_M \end{array}\right]Z_L$$
$$+\frac{c_P}{\psi_P}\left[\begin{array}{c} -4\left(1+\frac{1-\eta}{\psi_P+(\lambda-1)(1-\eta)}\right)\frac{(1-\eta)^2}{(\psi_P+(\lambda-1)(1-\eta))^2}Z_M \\ +\frac{(1-\eta)^2}{(\psi_A+(\lambda-1)(1-\eta))^2}\left(1+\frac{1-\eta}{\psi_P+(\lambda-1)(1-\eta)}\right)Z_L \\ +\left(1+\frac{1-\eta}{\psi_A+(\lambda-1)(1-\eta)}\right)\frac{(1-\eta)^2}{(\psi_P+(\lambda-1)(1-\eta))^2}Z_L+R_0\frac{(1-\eta)^2}{(\psi_P+(\lambda-1)(1-\eta))^2} \end{array}\right]Z_M,$$

or equivalently,

$$0 > \frac{c_A}{\psi_A}Z_L\left[\begin{array}{c} \frac{(1-\eta)^2}{(\psi_A+(\lambda-1)(1-\eta))^2}\left(-\left(1+\frac{1-\eta}{\psi_A+(\lambda-1)(1-\eta)}\right)Z_L+\left(1+\frac{1-\eta}{\psi_P+(\lambda-1)(1-\eta)}\right)Z_M\right)+ \\ \left(1+\frac{1-\eta}{\psi_A+(\lambda-1)(1-\eta)}\right)\left(-\frac{(1-\eta)^2}{(\psi_A+(\lambda-1)(1-\eta))^2}Z_L+\frac{(1-\eta)^2}{(\psi_P+(\lambda-1)(1-\eta))^2}Z_M\right) \end{array}\right] \tag{78}$$
$$+\frac{c_P}{\psi_P}Z_M\left[\begin{array}{c} \frac{(1-\eta)^2}{(\psi_P+(\lambda-1)(1-\eta))^2}\left(-2\left(1+\frac{1-\eta}{\psi_P+(\lambda-1)(1-\eta)}\right)Z_M+R_0+\left(1+\frac{1-\eta}{\psi_A+(\lambda-1)(1-\eta)}\right)Z_L\right) \\ +\left(1+\frac{1-\eta}{\psi_P+(\lambda-1)(1-\eta)}\right)\left(-2\frac{(1-\eta)^2}{(\psi_P+(\lambda-1)(1-\eta))^2}Z_M+\frac{(1-\eta)^2}{(\psi_A+(\lambda-1)(1-\eta))^2}Z_L\right) \end{array}\right],$$

By (76)-(77), the first line in each of the square brackets in (78) is nonpositive. Therefore, to prove that (78) holds, it is sufficient to show that

$$0 > \frac{c_A}{\psi_A}\left[\begin{array}{c} -\left(1+\frac{1-\eta}{\psi_A+(\lambda-1)(1-\eta)}\right)\frac{1}{(\psi_A+(\lambda-1)(1-\eta))^2}Z_L \\ +\left(1+\frac{1-\eta}{\psi_A+(\lambda-1)(1-\eta)}\right)\frac{1}{(\psi_P+(\lambda-1)(1-\eta))^2}Z_M \end{array}\right]Z_L$$
$$+\frac{c_P}{\psi_P}\left[\begin{array}{c} -2\left(1+\frac{1-\eta}{\psi_P+(\lambda-1)(1-\eta)}\right)\frac{1}{(\psi_P+(\lambda-1)(1-\eta))^2}Z_M \\ +\frac{1}{(\psi_A+(\lambda-1)(1-\eta))^2}\left(1+\frac{1-\eta}{\psi_P+(\lambda-1)(1-\eta)}\right)Z_L \end{array}\right]Z_M$$
$$\Leftrightarrow\ 0 > \frac{(\psi_A+(\lambda-1)(1-\eta))^2}{(\psi_P+(\lambda-1)(1-\eta))^2}\left[\begin{array}{c} \frac{c_A}{c_P}\frac{\psi_P}{\psi_A}\frac{\psi_A+\lambda(1-\eta)}{\psi_A+(\lambda-1)(1-\eta)}Z_L \\ -2\frac{\psi_P+\lambda(1-\eta)}{\psi_P+(\lambda-1)(1-\eta)}Z_M \end{array}\right]Z_M$$
$$+\left[\frac{\psi_P+\lambda(1-\eta)}{\psi_P+(\lambda-1)(1-\eta)}Z_M-\frac{c_A}{c_P}\frac{\psi_P}{\psi_A}\frac{\psi_A+\lambda(1-\eta)}{\psi_A+(\lambda-1)(1-\eta)}Z_L\right]Z_L$$

Letting

$$x \equiv \frac{\psi_A+\lambda(1-\eta)}{\psi_A+(\lambda-1)(1-\eta)}Z_L,$$
$$y \equiv \frac{\psi_P+\lambda(1-\eta)}{\psi_P+(\lambda-1)(1-\eta)}Z_M,$$

this condition can be expressed as

$$\begin{aligned} 0 &> \frac{(\psi_A+(\lambda-1)(1-\eta))^2}{(\psi_P+(\lambda-1)(1-\eta))^2}\left[\frac{c_A}{c_P}\frac{\psi_P}{\psi_A}x-2y\right]Z_M+\left[y-\frac{c_A}{c_P}\frac{\psi_P}{\psi_A}x\right]Z_L \\ &\Leftrightarrow\ 0>\frac{(\psi_A+(\lambda-1)(1-\eta))^2}{(\psi_P+(\lambda-1)(1-\eta))^2}[x-2y]\frac{Z_M}{Z_L}+[y-x] \\ &\quad+\left(\frac{c_A}{c_P}\frac{\psi_P}{\psi_A}-1\right)\left[\frac{(\psi_A+(\lambda-1)(1-\eta))(\psi_A+\lambda(1-\eta))}{(\psi_P+(\lambda-1)(1-\eta))(\psi_P+\lambda(1-\eta))}y-x\right]. \end{aligned}$$

Denoting $a\equiv x-y$ and $b\equiv 2y-x$, this condition becomes

$$\frac{(\psi_A+(\lambda-1)(1-\eta))^2}{(\psi_P+(\lambda-1)(1-\eta))^2}\frac{Z_M}{Z_L}b+a>\left(\frac{c_A}{c_P}\frac{\psi_P}{\psi_A}-1\right)\left[\frac{(\psi_A+(\lambda-1)(1-\eta))(\psi_A+\lambda(1-\eta))}{(\psi_P+(\lambda-1)(1-\eta))(\psi_P+\lambda(1-\eta))}y-x\right]. \tag{79}$$

Note that (76) implies that $a\geq 0$ and (77) implies that $b>0$, and hence the left-hand side in (79) is always positive.

Suppose that $|c_P-c_A|$ is sufficiently small. Then, by continuity of λ in c_A and c_P, it is sufficient to show that (79) holds if $c_P=c_A$. Therefore, there are three cases to consider. First, suppose that $\psi_P\geq\psi_A$. Then

$$\frac{(\psi_A+(\lambda-1)(1-\eta))(\psi_A+\lambda(1-\eta))}{(\psi_P+(\lambda-1)(1-\eta))(\psi_P+\lambda(1-\eta))}y-x\leq y-x=-a\leq 0,$$

and hence the right-hand side in (79) is nonpositive, concluding the argument. Second, suppose that $\psi_P<\psi_A$ and

$$\frac{(\psi_A+(\lambda-1)(1-\eta))(\psi_A+\lambda(1-\eta))}{(\psi_P+(\lambda-1)(1-\eta))(\psi_P+\lambda(1-\eta))}y-x\geq 0.$$

Then, the right-hand side in (79) is nonpositive, concluding the argument. Third, suppose that $\psi_P<\psi_A$ and

$$\frac{(\psi_A+(\lambda-1)(1-\eta))(\psi_A+\lambda(1-\eta))}{(\psi_P+(\lambda-1)(1-\eta))(\psi_P+\lambda(1-\eta))}y-x<0.$$

Then,

$$\begin{aligned} a &= x-y>x-\frac{(\psi_A+(\lambda-1)(1-\eta))(\psi_A+\lambda(1-\eta))}{(\psi_P+(\lambda-1)(1-\eta))(\psi_P+\lambda(1-\eta))}y \\ &\geq \left(1-\frac{\psi_P}{\psi_A}\right)\left[x-\frac{(\psi_A+(\lambda-1)(1-\eta))(\psi_A+\lambda(1-\eta))}{(\psi_P+(\lambda-1)(1-\eta))(\psi_P+\lambda(1-\eta))}y\right] \\ &= \left(\frac{\psi_P}{\psi_A}-1\right)\left[\frac{(\psi_A+(\lambda-1)(1-\eta))(\psi_A+\lambda(1-\eta))}{(\psi_P+(\lambda-1)(1-\eta))(\psi_P+\lambda(1-\eta))}y-x\right], \end{aligned}$$

which implies that (79) is satisfied since $b > 0$.

Next, suppose that $\psi_A \geq \psi_P$ and $c_A \leq \frac{\psi_A}{\psi_P} c_P$. There are two cases to consider. First, suppose that

$$\frac{(\psi_A + (\lambda - 1)(1-\eta))(\psi_A + \lambda(1-\eta))}{(\psi_P + (\lambda - 1)(1-\eta))(\psi_P + \lambda(1-\eta))} y - x \geq 0.$$

Then, the right-hand side in (79) is nonpositive, concluding the argument. Second, suppose that

$$\frac{(\psi_A + (\lambda - 1)(1-\eta))(\psi_A + \lambda(1-\eta))}{(\psi_P + (\lambda - 1)(1-\eta))(\psi_P + \lambda(1-\eta))} y - x < 0.$$

Then,

$$\begin{aligned} a &= x - y \geq x - \frac{(\psi_A + (\lambda - 1)(1-\eta))(\psi_A + \lambda(1-\eta))}{(\psi_P + (\lambda - 1)(1-\eta))(\psi_P + \lambda(1-\eta))} y \\ &> \left(1 - \frac{c_A}{c_P}\frac{\psi_P}{\psi_A}\right)\left[x - \frac{(\psi_A + (\lambda - 1)(1-\eta))(\psi_A + \lambda(1-\eta))}{(\psi_P + (\lambda - 1)(1-\eta))(\psi_P + \lambda(1-\eta))} y\right] \\ &= \left(\frac{c_A}{c_P}\frac{\psi_P}{\psi_A} - 1\right)\left[\frac{(\psi_A + (\lambda - 1)(1-\eta))(\psi_A + \lambda(1-\eta))}{(\psi_P + (\lambda - 1)(1-\eta))(\psi_P + \lambda(1-\eta))} y - x\right]. \end{aligned}$$

This implies that (79) is satisfied since $b > 0$, concluding the first step of the proof. ■

Lemma 3 (diversification across L-stocks) *If the cost function is quadratic, the active fund finds it optimal to diversify across L-stocks and invest the same amount in each L-stock.*

Proof of Lemma 3. Consider the problem of the active fund manager subject to investing only in L-firms. What will be the price that an active fund manager needs to pay to acquire x_{Aj} shares of firm j? Since the holdings of the passive fund are fixed by her assets under management and the requirement to hold a value-weighted portfolio, competition among liquidity investors means that the relationship between x_{Aj} and P_j must satisfy:

$$P_j = R_0 + c_A'^{-1}(f_A x_{Aj}) + c_P'^{-1}(f_P x_P) - Z_j.$$

Therefore, to acquire x_{Aj} shares, the active fund manager must pay

$$x_{Aj}\left(R_0 + c_A'^{-1}(f_A x_{Aj}) + c_P'^{-1}(f_P x_P) - Z_j\right).$$

Her cost of effort for firm j is $c_A\left(c_A'^{-1}(f_A x_{Aj})\right)$. Thus, the portfolio optimization problem of the active fund manager is:

$$\int \left[f_A x_{Aj}\left(R_0 + c_A'^{-1}(f_A x_{Aj}) + c_P'^{-1}(f_P x_P)\right) - c_A\left(c_A'^{-1}(f_A x_{Aj})\right)\right] dj$$

subject to

$$\int x_{Aj}\left(R_0 + c_A'^{-1}\left(f_A x_{Aj}\right) + c_P'^{-1}\left(f_P x_P\right) - Z_j\right) dj = W_A.$$

Let $F(t) = \max_e \{te - c_A(e)\}$. Then, we can re-write this optimization problem as:

$$\begin{array}{c} \int \left[f_A x_{Aj}\left(R_0 + c_P'^{-1}\left(f_P x_P\right)\right) + F\left(f_A x_{Aj}\right)\right] dj \\ \text{s.t. } \int x_{Aj}\left(R_0 + c_A'^{-1}\left(f_A x_{Aj}\right) + c_P'^{-1}\left(f_P x_P\right) - Z_j\right) dj = W_A. \end{array}$$

Let μ_0 denote the Lagrange multiplier of the budget constraint and μ_j denote the Lagrange multiplier of the no short-sale constraint for stock j. Then, the optimal portfolio choice solves

$$\begin{array}{c} \max_{x_{Aj},\mu_0,\mu} \int \left[f_A x_{Aj}\left(R_0 + c_P'^{-1}\left(f_P x_P\right)\right) + F\left(f_A x_{Aj}\right)\right] dj \\ +\mu_0\left(W_A - \int x_{Aj}\left(R_0 + c_A'^{-1}\left(f_A x_{Aj}\right) + c_P'^{-1}\left(f_P x_P\right) - Z_j\right) dj\right) + \int \mu_j x_{Aj} dj. \end{array}$$

The first-order condition with respect to x_{Aj} is (applying the envelope theorem to $F(\cdot)$):

$$f_A\left(R_0 + c_P'^{-1}\left(f_P x_P\right) + c_A'^{-1}\left(f_A x_{Aj}\right)\right) - \mu_0 \left(\begin{array}{c} R_0 + c_A'^{-1}\left(f_A x_{Aj}\right) + c_P'^{-1}\left(f_P x_P\right) \\ -Z_j + x_{Aj}\left[\frac{dc_A'^{-1}\left(f_A x_{Aj}\right)}{dx_{Aj}}\right] \end{array}\right) + \mu_j = 0$$

$$\Leftrightarrow \left(f_A - \mu_0\right)\left(R_0 + c_P'^{-1}\left(f_P x_P\right) + c_A'^{-1}\left(f_A x_{Aj}\right)\right) - \mu_0\left(-Z_j + \frac{f_A x_{Aj}}{c_A''\left(c_A'^{-1}\left(f_A x_{Aj}\right)\right)}\right) + \mu_j = 0.$$

Two cases are possible.

(1) First, $\mu_j = 0 \; \forall j$. Then, $x_{Aj} = x_A$ for all j. Indeed: we have exactly the same equation on all $f_A x_{Aj}$:

$$\begin{aligned} \left(f_A - \mu_0\right)\left(R_0 + c_P'^{-1}\left(f_P x_P\right) + c_A'^{-1}\left(f_A x_{Aj}\right)\right) &= \mu_0\left(-Z_j + \frac{f_A x_{Aj}}{c_A''\left(c_A'^{-1}\left(f_A x_{Aj}\right)\right)}\right) \Leftrightarrow \\ f_A R_j &= \mu_0\left(R_j - Z_j + \frac{f_A x_{Aj}}{c_A''\left(c_A'^{-1}\left(f_A x_{Aj}\right)\right)}\right). \end{aligned}$$

1. It follows that $\mu_0 > 0$ since both the left-hand-side and the term in brackets are strictly positive.

2. In addition, we show that for a quadratic cost function, $f_A < 2\mu_0$. Indeed, suppose not, and $f_A \geq 2\mu_0$. Then $f_A\left(P_j + Z_j\right) \geq 2\mu_0\left(P_j + Z_j\right)$, so using $P_J = R_j - Z_j$, we have

$$\begin{aligned} \mu_0\left(P_j + \frac{f_A x_{Aj}}{c_A''\left(e_{Aj}\right)}\right) &\geq 2\lambda\left(P_j + Z_j\right) \Leftrightarrow R_j - Z_j \leq -2Z_j + \frac{f_A x_{Aj}}{c_A''\left(e_{Aj}\right)} \Leftrightarrow \\ R_0 + c_P'^{-1}\left(f_P x_P\right) + e_{Aj} &\leq -Z_j + \frac{f_A x_{Aj}}{c_A''\left(e_{Aj}\right)}. \end{aligned}$$

Since $R_0 + c_P'^{-1}\left(f_P x_P\right) > 0 > -Z_j$, then to prove the above statement by contradiction, it is sufficient to show that $e_{Aj} \geq \frac{f_A x_{Aj}}{c''_A(e_{Aj})}$.

Consider $c_A(e) = c_A e^\alpha, \alpha > 1$. Then e solves $\arg\max\{te - c_A e^\alpha\} \Rightarrow\ t = c_A \alpha e^{\alpha-1} \Leftrightarrow e = \left(\frac{t}{\alpha c_A}\right)^{\frac{1}{\alpha-1}}$. Then

$$\frac{f_A x_{Aj}}{c''_A(e_{Aj})} = \frac{t}{c''_A(e_{Aj})} = \frac{t}{c_A \alpha(\alpha-1)e^{\alpha-2}} = \frac{c_A \alpha e^{\alpha-1}}{c_A \alpha(\alpha-1)e^{\alpha-2}} = \frac{e}{\alpha-1}$$

and comparing to e, we conclude that $e \geq \frac{f_A x_{Aj}}{c''_A(e_{Aj})} \Leftrightarrow e \geq \frac{e}{\alpha-1} \Leftrightarrow \alpha - 1 \geq 1 \Leftrightarrow \alpha \geq 2$.

Hence, when $\alpha \geq 2$, then indeed, $f_A < 2\mu_0$. To check that this is the optimum, we need to verify the second-order condition. The second-order derivative with respect to x_{Aj} is:

$$(f_A - \mu_0)\frac{f_A}{c''_A(e_{Aj})} - \mu_0 \frac{f_A}{c''_A(e_{Aj})} - \mu_0 x_{Aj}\frac{d^2 e_{Aj}}{dx^2_{Aj}}.$$

Since the Hessian matrix is a diagonal (i.e., the cross-partial derivative w.r.to $x_{Aj}x_{Ak}$ is zero), the second-order condition is simply

$$(f_A - 2\mu_0)\frac{f_A}{c''_A(e_{Aj})} - \mu_0 x_{Aj}\frac{d^2 e_{Aj}}{dx^2_{Aj}} < 0. \tag{80}$$

As shown above, $f_A - 2\mu_0 < 0$ for $\alpha \geq 2$, and hence the first term is negative. Since $\mu_0 > 0$, a sufficient condition for (80) to hold is $\frac{d^2 e_{Aj}}{dx^2_{Aj}} \geq 0$.

For a power cost function, note that $e(t) \sim t^{\frac{1}{\alpha-1}}$ and hence $e''(t) \geq 0 \Leftrightarrow \left(\frac{1}{\alpha-1}\right)\left(\frac{1}{\alpha-1} - 1\right) \geq 0 \Leftrightarrow 2 - \alpha \geq 0 \Leftrightarrow \alpha \leq 2$. Hence, when $\alpha = 2$, we have proved that the second-order condition (globally) is satisfied.

(2) Second, $\mu_j = 0$ for some j and $\mu_j > 0$ for other j. For the latter firms, $x_{Aj} = 0$, so μ_j (using FOC) satisfies:

$$\begin{aligned}(f_A - \mu_0)\left(R_0 + c'^{-1}_P(f_P x_P)\right) + \lambda Z_j + \mu_j = 0\\ \Leftrightarrow \mu_j = -\lambda Z_j - (f_A - \mu_0)\left(R_0 + c'^{-1}_P(f_P x_P)\right),\end{aligned}$$

which is the same for all firms. For the former firms, x_{Aj} satisfies

$$(f_A - \mu_0)\left(R_0 + c'^{-1}_P(f_P x_P) + c'^{-1}_A(f_A x_{Aj})\right) - \mu_0\left(-Z_j + x_{Aj}\left[\frac{dc'^{-1}_A(f_A x_{Aj})}{dx_{Aj}}\right]\right) = 0,$$

which is the same as the first-order condition from the first case. Hence, the fund will always invest a symmetric amount in whatever subset of stocks it invests in. ∎

Lemma 4 (sufficient conditions for not investing in H-stocks) *(i) For a given set of parameters and the conjectured equilibrium effort levels e_{AL}, e_P, the active fund does* *not find it optimal to deviate to investing in H-stocks if* $Z_L - Z_H > e_{AL}\left(1 + \frac{Z_H}{R_0 + e_P}\right)$.

(*ii*) *Suppose* $\frac{Z_M}{Z_L} > \frac{\xi_A \xi_P + \xi_A - \xi_P}{\xi_P^2}$, *where*

$$\xi_A \equiv \xi_A(\lambda_{\max}) = 1 + \frac{1}{\frac{\psi_A}{1-\eta} + \frac{R_0}{R_0 - Z_L} - \psi_A - 1}, \tag{81}$$

$$\xi_P \equiv \xi_P(\lambda_{\max}) = 1 + \frac{1}{\frac{\psi_P}{1-\eta} + \frac{R_0}{R_0 - Z_L} - \psi_A - 1}. \tag{82}$$

Then, given the equilibrium characterized by Proposition 1, the active fund does not find it optimal to deviate to investing in H-stocks.

(*iii*) *Suppose*

$$Z_L < \frac{R_0}{\frac{1-\eta}{\psi_A} + \left(1 + \frac{1-\eta}{\psi_A}\right)\frac{Z_H}{R_0}}.$$

Then, given any equilibrium characterized by Lemma 1, the active fund does not find it optimal to deviate to investing in H-stocks.

Proof of Lemma 4.

Proof of part (*i*)**.** Consider the problem of the active fund manager. Since the holdings of the passive fund are fixed by her assets under management and the requirement to hold a value-weighted portfolio, competition among liquidity investors means that the relationship between x_{Aj} and P_j must satisfy:

$$P_j = R_0 + c_A'^{-1}(f_A x_{Aj}) + c_P'^{-1}(f_P x_P) - Z_j.$$

To acquire x_{Aj} shares, the active fund manager must pay

$$x_{Aj}\left(R_0 + c_A'^{-1}(f_A x_{Aj}) + c_P'^{-1}(f_P x_P) - Z_j\right).$$

Her cost of effort for firm j is $c_A\left(c_A'^{-1}(f_A x_{Aj})\right)$. Thus, the portfolio optimization problem of the active fund manager is:

$$\int \left[f_A x_{Aj}\left(R_0 + c_A'^{-1}(f_A x_{Aj}) + c_P'^{-1}(f_P x_P)\right) - c_A\left(c_A'^{-1}(f_A x_{Aj})\right)\right] dj$$

subject to

$$\int x_{Aj}\left(R_0 + c_A'^{-1}(f_A x_{Aj}) + c_P'^{-1}(f_P x_P) - Z_j\right) dj = W_A.$$

Let $F(t) = \max_e \{te - c_A(e)\}$. Then, we can re-write this optimization problem as:

$$\begin{array}{c} \int \left[f_A x_{Aj} \left(R_0 + c_P'^{-1}(f_P x_P)\right) + F(f_A x_{Aj})\right] dj \\ \text{s.t.} \ \int x_{Aj} \left(R_0 + c_A'^{-1}(f_A x_{Aj}) + c_P'^{-1}(f_P x_P) - Z_j\right) dj = W_A \end{array}$$

Consider the solution in which $x_{Aj} = 0$ for all H-stocks. As shown in Section 2 of this document, for a quadratic cost function, we then have $x_{Aj} = x_{AL} = \frac{2W_A}{P_L}$ for all L-stocks.

We next find sufficient conditions for a small deviation to investing in H-stocks to not be profitable. Consider a deviation to $x_{Aj} = x_{AL} - \delta$ for L-stocks and $x_{Aj} = \varepsilon$ for H-stocks such that budget constraint is satisfied. Then the budget constraint implies:

$$\frac{1}{2}\varepsilon P_{H,new} + \frac{1}{2}(x_{AL} - \delta) P_{L,new} = \frac{1}{2} x_{AL} P_{L,old},$$

where

$$\begin{aligned} P_{L,old} &= R_0 + c_A'^{-1}(f_A x_{AL}) + c_P'^{-1}(f_P x_P) - Z_L, \\ P_{L,new} &= R_0 + c_A'^{-1}(f_A x_{AL} - f_A\delta) + c_P'^{-1}(f_P x_P) - Z_L, \\ P_{H,new} &= R_0 + c_A'^{-1}(f_A\varepsilon) + c_P'^{-1}(f_P x_P) - Z_H. \end{aligned}$$

Since $c_A'^{-1}(y) = \frac{y}{c_A}$, and denoting $e_P = c_P'^{-1}(f_P x_P)$, the budget constraint is equivalent to

$$\varepsilon\left(R_0 + c_A'^{-1}(f_A\varepsilon) + e_P - Z_H\right) = \delta\left(R_0 + c_A'^{-1}(f_A x_{AL} - f_A\delta) + e_P - Z_L\right) + x_{AL}\frac{f_A\delta}{c_A}$$

Differentiating this w.r.to ε and taking the limit $\varepsilon \to 0, \delta \to 0$, we get:

$$\begin{aligned} R_0 + \frac{f_A\varepsilon}{c_A} + e_P - Z_H + \varepsilon\left[\frac{d}{d\varepsilon}\frac{f_A\varepsilon}{c_A}\right] &= \frac{d\delta}{d\varepsilon}\left[R_0 + \frac{f_A x_{AL} - f_A\delta}{c_A} + e_P - Z_L + x_{AL}\frac{f_A}{c_A} + \delta\frac{d}{d\delta}\frac{f_A x_{AL} - f_A\delta}{c_A}\right] \\ \Leftrightarrow \ \frac{d\delta}{d\varepsilon} &= \frac{R_0 + e_P - Z_H + 2\frac{f_A\varepsilon}{c_A}}{R_0 + \frac{2f_A x_{AL} - 2f_A\delta}{c_A} + e_P - Z_L}. \end{aligned}$$

The payoff Π from this deviation satisfies:

$$\begin{aligned} 2\Pi &= 2\int \left[f_A x_{Aj}(R_0 + e_P) + F(f_A x_{Aj})\right] dj \qquad (83) \\ &= f_A(x_{AL} - \delta)(R_0 + e_P) + F(f_A(x_{AL} - \delta)) + f_A\varepsilon(R_0 + e_P) + F(f_A\varepsilon), \end{aligned}$$

where $F(t) = \max_e \{te - c_A(e)\}$.

Note that $\frac{dF(f_A(x_{AL}-\delta))}{d\delta} = -\frac{f_A^2(x_{AL}-\delta)}{c_A}$ because by envelope theorem $F_\delta' = F_t'\frac{dt}{d\delta} = [t = f_A(x_{AL} - \delta)] = -f_A F_t' = -f_A c_A'^{-1}[f_A(x_{AL} - \delta)] = -\frac{f_A^2(x_{AL}-\delta)}{c_A}$.

Similarly, $\frac{dF(f_A\varepsilon)}{d\varepsilon} = F_t'\frac{dt}{d\varepsilon} = [t = f_A\varepsilon] = f_A F_t' = f_A c_A'^{-1}[f_A\varepsilon] = \frac{f_A^2\varepsilon}{c_A}$.

Hence, differentiating (83) w.r.to ε, we have:

$$\begin{aligned} 2\frac{d\Pi}{d\varepsilon} &= \frac{d}{d\delta}\left[f_A\left(x_{AL}-\delta\right)\left(R_0+e_P\right)+F\left(f_A\left(x_{AL}-\delta\right)\right)\right]\frac{d\delta}{d\varepsilon}+\frac{d}{d\varepsilon}\left[f_A\varepsilon\left(R_0+e_P\right)+F\left(f_A\varepsilon\right)\right] \\ &= \frac{R_0+e_P-Z_H+2\frac{f_A\varepsilon}{c_A}}{R_0+\frac{2f_Ax_{AL}-2f_A\delta}{c_A}+e_P-Z_L}\left[-f_A\left(R_0+e_P\right)-\frac{f_A^2\left(x_{AL}-\delta\right)}{c_A}\right]+\left[f_A\left(R_0+e_P\right)+\frac{f_A^2\varepsilon}{c_A}\right]. \end{aligned}$$

Hence, $\frac{d\Pi}{d\varepsilon}<0$ if and only if

$$R_0+e_P+\frac{f_A\varepsilon}{c_A}<\frac{R_0+e_P-Z_H+2\frac{f_A\varepsilon}{c_A}}{R_0+\frac{2f_Ax_{AL}-2f_A\delta}{c_A}+e_P-Z_L}\left[R_0+e_P+\frac{f_A\left(x_{AL}-\delta\right)}{c_A}\right]$$

and taking the limit $\varepsilon\to 0,\delta\to 0$,

$$R_0+e_P<\frac{R_0+e_P-Z_H}{R_0+\frac{2f_Ax_{AL}}{c_A}+e_P-Z_L}\left[R_0+e_P+\frac{f_Ax_{AL}}{c_A}\right]$$

Denoting $r_P\equiv R_0+e_P$,

$$\begin{aligned} \frac{d\Pi}{d\varepsilon} &< 0\Leftrightarrow r_P\left[r_P-Z_L+2\frac{f_Ax_{AL}}{c_A}\right]<\left(r_P-Z_H\right)\left[r_P+\frac{f_Ax_{AL}}{c_A}\right] \\ &\Leftrightarrow 0<r_P\left(Z_L-Z_H-\frac{f_Ax_{AL}}{c_A}\right)-Z_H\frac{f_Ax_{AL}}{c_A}. \\ &\Leftrightarrow 0<\left(R_0+e_P\right)\left(Z_L-Z_H-e_{AL}\right)-Z_He_{AL} \\ &\Leftrightarrow Z_L-Z_H>e_{AL}\left(1+\frac{Z_H}{R_0+e_P}\right), \end{aligned} \tag{84}$$

which proves part (i).

Proof of part (ii)**.** To prove this part, we show that the conditions in part (ii) are sufficient for (84) to hold. We reformulate (84) in terms of $Z_M=\frac{Z_H+Z_L}{2}$ and Z_L and use (73)-(74):

$$\begin{aligned} -2Z_M+2Z_L &> e_{AL}\left(1+\frac{2Z_M-Z_L}{R_0+e_P}\right)=2\left(R_L-R_M\right)\left(1+\frac{2Z_M-Z_L}{R_0+e_P}\right)\Leftrightarrow \\ -Z_M+Z_L &> \left(\xi_A\left(\lambda\right)Z_L-\xi_P\left(\lambda\right)Z_M\right)\left(1-\frac{Z_L-2Z_M}{R_0+e_P}\right). \end{aligned}$$

Plugging in $e_P=2\xi_P\left(\lambda\right)Z_M-\xi_A\left(\lambda\right)Z_L-R_0$, we get

$$\left(2\xi_P\left(\lambda\right)Z_M-\xi_A\left(\lambda\right)Z_L\right)\left(Z_L-Z_M\right)>\left(\xi_A\left(\lambda\right)Z_L-\xi_P\left(\lambda\right)Z_M\right)\left(2\xi_P\left(\lambda\right)Z_M+2Z_M-\xi_A\left(\lambda\right)Z_L-Z_L\right).$$

Simplifying and rearranging, this is equivalent to

$$\left(\xi_A(\lambda) Z_L - \xi_P(\lambda) Z_M\right)^2 + \xi_P^2(\lambda) Z_M^2 + Z_L Z_M\left(\xi_P(\lambda) - \xi_A(\lambda) - \xi_A(\lambda)\xi_P(\lambda)\right) > 0$$

Since the first term is non-negative, a sufficient condition is that the sum of the second and third term is strictly positive or, equivalently,

$$\frac{Z_M}{Z_L} > \frac{\xi_A(\lambda)\xi_P(\lambda) + \xi_A(\lambda) - \xi_P(\lambda)}{\xi_P^2(\lambda)}. \tag{85}$$

We next show that the right-hand side is increasing in λ. Indeed, denote $L_i \equiv \frac{\psi_i}{1-\eta} + \lambda - 1$, where $L_A \geq L_P$, and notice that

$$\begin{aligned}
\left(\frac{\xi_A(\lambda)\xi_P(\lambda) + \xi_A(\lambda) - \xi_P(\lambda)}{\xi_P^2(\lambda)}\right)' &\geq 0 \Leftrightarrow \\
\xi_A'(\lambda)\xi_P(\lambda)(\xi_P(\lambda)+1) &\geq \xi_P'(\lambda)\left[\xi_A(\lambda)\xi_P(\lambda) + 2\xi_A(\lambda) - \xi_P(\lambda)\right] \Leftrightarrow \\
\frac{-1}{L_A^2}\left(1+\frac{1}{L_P}\right)\left(2+\frac{1}{L_P}\right) &\geq \frac{-1}{L_P^2}\left[\left(1+\frac{1}{L_A}\right)\left(1+\frac{1}{L_P}\right) + 2 + \frac{2}{L_A} - 1 - \frac{1}{L_P}\right] \Leftrightarrow \\
L_A\left[2L_A L_P + 3L_P + 1\right] &\geq L_P\left(2L_P^2 + 3L_P + 1\right).
\end{aligned}$$

The last inequality automatically follows from the fact that $L_A \geq L_P$. Hence, if (85) is satisfied for the largest possible λ, i.e., $\lambda_{\max}$ from Lemma 8, then it is satisfied for any possible λ. This completes the proof of part (ii).

Proof of part (iii). By Lemma 1, there are two cases to consider: the equilibrium where only the passive funds raises positive AUM, and the equilibrium where only the active fund raises positive AUM. Note that since the arguments made in part (i) apply to these equilibria as well, it is sufficient to show that e_{AL}, e_P satisfy (84). First, suppose that the equilibrium is as described by part (i) of Lemma 1. Then, $x_{AL} = 0$ implies that the active fund exerts no effort, and hence $e_{AL} = 0$, so (84) is satisfied. Second, suppose that the equilibrium is as described by part (ii) of Lemma 1. Then, $\lambda = 1$ and $R_L = \left(1 + \frac{1-\eta}{\psi_A}\right) Z_L$ in equilibrium. Combining with $e_P = 0$ (due to $x_P = 0$) and $e_{AL} = R_L - R_0$, (84) is equivalent to

$$Z_L - Z_H > \left(\left(1 + \frac{1-\eta}{\psi_A}\right) Z_L - R_0\right)\left(1 + \frac{Z_H}{R_0}\right) \Leftrightarrow Z_L < \frac{R_0}{\frac{1-\eta}{\psi_A} + \left(1 + \frac{1-\eta}{\psi_A}\right)\frac{Z_H}{R_0}},$$

which completes the proof. ■

Lemma 5 (sufficient conditions for not investing in the outside asset) (i) *For a given set of parameters and the conjectured equilibrium payoffs R_L, R_M, the active fund does not find it optimal to deviate to investing in the outside asset if $2(R_L - R_M) < Z_L$.*

(*ii*) *Suppose* $\frac{Z_M}{Z_L} > \max\{0.64, \frac{\frac{1}{2}+\frac{1-\eta}{\psi_A}}{1+\frac{1-\eta}{\psi_P}}\}$. *Then, given the equilibrium characterized by Proposition 1, the active fund does not find it optimal to deviate to investing in the outside asset.*

(*iii*) *Suppose* $Z_L < \frac{\psi_A}{1-\eta}R_0$. *Then, given any equilibrium characterized by Lemma 1, the active fund does not find it optimal to deviate to investing in the outside asset.*

Proof of Lemma 5. Proof of part (i). Repeating the arguments of the proof of Lemma 4, let us consider the equilibrium in which $x_{Aj} = 0$ for all H-stocks and $x_{Aj} = x_{AL} = \frac{2W_A}{P_L}$ for all L-stocks. We next find sufficient conditions for a small deviation to investing in the outside asset to not be profitable. Consider a deviation to $x_{Aj} = x_{AL} - \delta$ for L-stocks and ε for the outside asset such that budget constraint is satisfied. Let P_{old} and P_{new} be the price of L-stocks before and after the deviation, where

$$\begin{aligned} P_{old} &= R_0 + c_A'^{-1}\left(f_A x_{AL}\right) + c_P'^{-1}\left(f_P x_P\right) - Z_L \\ P_{new} &= R_0 + c_A'^{-1}\left(f_A x_{AL} - f_A\delta\right) + c_P'^{-1}\left(f_P x_P\right) - Z_L \end{aligned}$$

Then the budget constraint implies:

$$\begin{aligned} \varepsilon + \frac{1}{2}\left(x_{AL} - \delta\right) P_{new} &= \frac{1}{2}x_{AL}P_{old} \Leftrightarrow \varepsilon = \frac{\delta}{2}P_{new} + \frac{x_{AL}}{2}\left(P_{old} - P_{new}\right) \Leftrightarrow \\ 2\varepsilon &= \delta\left(R_0 + \frac{f_A x_{AL} - f_A\delta}{c_A} + c_P'^{-1}\left(f_P x_P\right) - Z_L\right) + x_{AL}\frac{f_A\delta}{c_A}. \end{aligned}$$

since $c_A(e) = \frac{c_A}{2}e^2; c_A'(e) = c_A e; c_A'^{-1}(y) = \frac{y}{c_A}$.

Differentiating this w.r.to ε and taking the limit $\varepsilon \to 0, \delta \to 0$, we get:

$$\begin{aligned} 2 &= \frac{d\delta}{d\varepsilon}\left[R_0 + \frac{f_A x_{AL} - f_A\delta}{c_A} + c_P'^{-1}\left(f_P x_P\right) - Z_L + \delta\frac{d}{d\delta}\frac{f_A x_{AL} - f_A\delta}{c_A} + \frac{x_{AL}f_A}{c_A}\right] \Leftrightarrow \\ \frac{d\delta}{d\varepsilon} &= \frac{2}{R_0 + \frac{2f_A x_{AL} - 2f_A\delta}{c_A} + c_P'^{-1}\left(f_P x_P\right) - Z_L} \end{aligned}$$

The payoff Π from this deviation satisfies:

$$\begin{aligned} 2\Pi &= 2\int\left[f_A x_{Aj}\left(R_0 + c_P'^{-1}\left(f_P x_P\right)\right) + F\left(f_A x_{Aj}\right)\right]dj + 2f_A\varepsilon \\ &= f_A\left(x_{AL} - \delta\right)\left(R_0 + c_P'^{-1}\left(f_P x_P\right)\right) + F\left(f_A\left(x_{AL} - \delta\right)\right) + 2f_A\varepsilon, \end{aligned} \tag{86}$$

where $F(t) = \max_e\{te - c_A(e)\}$. As in the proof of Lemma 4, $\frac{dF(f_A(x_{AL}-\delta))}{d\delta} = -\frac{f_A^2(x_{AL}-\delta)}{c_A}$.

Hence, differentiating (83) w.r.to ε, we have:

$$\begin{aligned} 2\frac{d\Pi}{d\varepsilon} &= \frac{d}{d\delta}\left[f_A\left(x_{AL}-\delta\right)\left(R_0+c_P'^{-1}\left(f_P x_P\right)\right)+F\left(f_A\left(x_{AL}-\delta\right)\right)\right]\frac{d\delta}{d\varepsilon}+2f_A \\ &= \frac{2}{R_0+\frac{2f_A x_{AL}-2f_A\delta}{c_A}+c_P'^{-1}\left(f_P x_P\right)-Z_L}\left[-f_A\left(R_0+c_P'^{-1}\left(f_P x_P\right)\right)-\frac{f_A^2\left(x_{AL}-\delta\right)}{c_A}\right]+2f_A. \end{aligned}$$

Hence, $\frac{d\Pi}{d\varepsilon}<0$ if and only if

$$1<\frac{1}{R_0+\frac{2f_A x_{AL}-2f_A\delta}{c_A}+c_P'^{-1}\left(f_P x_P\right)-Z_L}\left[R_0+c_P'^{-1}\left(f_P x_P\right)+\frac{f_A\left(x_{AL}-\delta\right)}{c_A}\right]$$

and taking the limit $\varepsilon\to 0,\delta\to 0$,

$$\begin{aligned} 1 &< \frac{1}{R_0+\frac{2f_A x_{AL}}{c_A}+c_P'^{-1}\left(f_P x_P\right)-Z_L}\left[R_0+c_P'^{-1}\left(f_P x_P\right)+\frac{f_A x_{AL}}{c_A}\right] \\ &\Leftrightarrow 2\left(R_L-R_M\right)<Z_L, \end{aligned} \tag{87}$$

which proves part (i).

Proof of part (ii)**.** To prove this part, we show that the conditions in part (ii) are sufficient for (87) to hold. Using $R_L=(1+\frac{1-\eta}{\psi_A+(\lambda-1)(1-\eta)})Z_L$, $R_M=(1+\frac{1-\eta}{\psi_P+(\lambda-1)(1-\eta)})Z_M$, (87) becomes

$$\begin{gathered} 2(1+\tfrac{1-\eta}{\psi_A+(\lambda-1)(1-\eta)})Z_L-2(1+\tfrac{1-\eta}{\psi_P+(\lambda-1)(1-\eta)})Z_M<Z_L \\ \Leftrightarrow \frac{Z_M}{Z_L}>\frac{\frac{1}{2}+\frac{1}{\frac{\psi_A}{1-\eta}+(\lambda-1)}}{1+\frac{1}{\frac{\psi_P}{1-\eta}+(\lambda-1)}}\equiv f\left(\lambda\right). \end{gathered}$$

Note that $f\left(\lambda\right)$ either decreases in λ or has a hump-shape. Indeed, denoting $\varphi_i\equiv\frac{\psi_i}{1-\eta}-1$, we can rewrite $f\left(\lambda\right)=\frac{0.5+\frac{1}{\varphi_A+\lambda}}{1+\frac{1}{\varphi_P+\lambda}}$, and

$$f'>0\Leftrightarrow-\frac{1}{\left(\varphi_A+\lambda\right)^2}\left(1+\frac{1}{\varphi_P+\lambda}\right)>\left(0.5+\frac{1}{\varphi_A+\lambda}\right)\frac{-1}{\left(\varphi_P+\lambda\right)^2}$$

Note that $\varphi_i+\lambda>0\Leftrightarrow\frac{\psi_i}{1-\eta}-1+\lambda>0$, which holds since $\lambda\geq 1$. Hence, multiplying by $\left(\varphi_A+\lambda\right)^2\left(\varphi_P+\lambda\right)^2$, we get

$$\begin{aligned} f' &> 0\Leftrightarrow-\left(\left(\varphi_P+\lambda\right)^2+\varphi_P+\lambda\right)>-\left(0.5\left(\varphi_A+\lambda\right)^2+\varphi_A+\lambda\right) \\ &\Leftrightarrow \lambda^2-2\lambda\left(\varphi_A-2\varphi_P\right)+\left(2\varphi_P^2-\varphi_A^2+2\varphi_P-2\varphi_A\right)<0. \end{aligned}$$

The discriminant, D, satisfies

$$\frac{D}{4} = 2\left(\varphi_A - \varphi_P\right)\left(\varphi_A - \varphi_P + 1\right).$$

Since $\psi_A > \psi_P$, we have $D > 0$, and hence $f' > 0 \Leftrightarrow \lambda_1 < \lambda < \lambda_2$, where

$$\lambda_{1,2} = \left(\varphi_A - \varphi_P\right) \mp \sqrt{2\left(\varphi_A - \varphi_P\right)\left(\varphi_A - \varphi_P + 1\right)}$$

Note that $\lambda_1 < 1$. Indeed,

$$\lambda_1 < 1 < 1 \Leftrightarrow \varphi_A - \varphi_P - 1 < \sqrt{2\left(\varphi_A - \varphi_P\right)\left(\varphi_A - \varphi_P + 1\right)}.$$

If $\varphi_A - \varphi_P - 1 < 0$, this is automatically satisfied, and if $\varphi_A - \varphi_P - 1 > 0$, this is equivalent to

$$\begin{aligned} \left(\varphi_A - \varphi_P - 1\right)^2 &< 2\left(\varphi_A - \varphi_P\right)\left(\varphi_A - \varphi_P + 1\right) \Leftrightarrow \\ 1 &< \left(\varphi_A - \varphi_P\right)^2 + 4\left(\varphi_A - \varphi_P\right), \end{aligned}$$

which holds because $\varphi_A - \varphi_P > 1$. Hence, we have two cases:

(1) First, if $\lambda_2 < 1$, then $f' < 0$ for all $\lambda \geq 1$. In this case, a sufficient condition for $\frac{Z_M}{Z_L} > f(\lambda)$ to hold for all λ is that $\frac{Z_M}{Z_L} > f(1) = \frac{\frac{1}{2} + \frac{1-\eta}{\psi_A}}{1 + \frac{1-\eta}{\psi_P}}$.

(2) Second, if $\lambda_2 > 1$, then f achieves its maximum at $\lambda = \lambda_2$. In this case, a sufficient condition for $\frac{Z_M}{Z_L} > f(\lambda)$ to hold for all λ is that $\frac{Z_M}{Z_L} > f(\lambda_2)$. We show that in this case, $f(\lambda_2) \leq 0.64$. Indeed, denote $\delta \equiv \varphi_A - \varphi_P$, and note that $\lambda_2 \geq 1 \Leftrightarrow \delta + \sqrt{2\delta(\delta + 1)} \geq 1 \Leftrightarrow \sqrt{2\delta(\delta + 1)} \geq 1 - \delta$. This, in turn, holds if either (1) $\delta \geq 1$ or (2) $\delta \leq 1$ and $2\delta^2 + 2\delta \geq 1 - 2\delta + \delta^2 \Leftrightarrow \delta \geq \sqrt{5} - 2$ (since $\delta \geq 0$). Combining these two conditions, $\lambda_2 \geq 1 \Leftrightarrow \delta = \varphi_A - \varphi_P \geq \varepsilon \equiv \sqrt{5} - 2$. If this is satisfied, then

$$f(\lambda_2) = \frac{0.5 + \frac{1}{\varphi_A + \lambda}}{1 + \frac{1}{\varphi_P + \lambda}} \leq \frac{0.5 + \frac{1}{\varphi_P + \varepsilon + \lambda}}{1 + \frac{1}{\varphi_P + \lambda}} = \frac{0.5 + \frac{1}{x + \varepsilon}}{1 + \frac{1}{x}} \equiv g(x), \tag{88}$$

where $x \equiv \varphi_P + \lambda = \frac{\psi_P}{1-\eta} - 1 + \lambda \geq 0$. Note that

$$\begin{aligned} g'(x) &\geq 0 \Leftrightarrow x^2 - 2\varepsilon x - (\varepsilon + 2) \leq 0 \\ x &\in \frac{2\varepsilon \pm \sqrt{4\varepsilon^2 + 4\varepsilon + 8}}{2} = \varepsilon \pm \sqrt{\varepsilon^2 + \varepsilon + 2} \end{aligned}$$

Since $x \geq 0$, this is equivalent to $x \leq \varepsilon + \sqrt{\varepsilon^2 + \varepsilon + 2} \Leftrightarrow \varphi_P + \lambda \leq \varepsilon + \sqrt{\varepsilon^2 + \varepsilon + 2}$

The two roots of the quadratic equation are $\varepsilon \pm \sqrt{\varepsilon^2 + \varepsilon + 2}$, and since $x \geq 0$, we have $g'(x) > 0 \Leftrightarrow x \leq \varepsilon + \sqrt{\varepsilon^2 + \varepsilon + 2}$. It follows that $g(x)$ achieves its maximum at $x^* = \varepsilon + \sqrt{\varepsilon^2 + \varepsilon + 2}$, where $g(x^*) < 0.64$. Hence, (88) implies that when $\lambda_2 \geq 1$, $f(\lambda_2) < 0.64$.

Combining the two cases, a sufficient condition for $\frac{Z_M}{Z_L} > f(\lambda)$ to hold for all λ is that $\frac{Z_M}{Z_L} > \max\{\frac{\frac{1}{2}+\frac{1-\eta}{\psi_A}}{1+\frac{1-\eta}{\psi_P}}, 0.64\}$, which completes the proof of part (ii).

Proof of part (iii)**.** By Lemma 1, there are two cases to consider: the equilibrium where only the passive funds raises positive AUM, and the equilibrium where only the active fund raises positive AUM. Note that since the arguments made in part (i) apply to these equilibria as well, it is sufficient to show that R_L, R_M satisfy (87). First, suppose that the equilibrium is as described by part (i) of Lemma 1. Then, $x_{AL} = 0$ implies that the active fund exerts no effort, and hence $e_{AL} = 0$, so $2(R_L - R_M) = 0 < Z_L$. Second, suppose that the equilibrium is as described by part (ii) of Lemma 1. Then, $\lambda = 1$ and $R_L = \left(1 + \frac{1-\eta}{\psi_A}\right) Z_L$ in equilibrium. Combining with $e_P = 0$ (due to $x_P = 0$) and $R_M = \frac{1}{2}R_L + \frac{1}{2}R_0$, (87) is equivalent to

$$\left(1 + \frac{1-\eta}{\psi_A}\right) Z_L - R_0 < Z_L \Leftrightarrow Z_L < \frac{\psi_A}{1-\eta} R_0,$$

which completes the proof. ■

Lemma 6 (positive assets under management) *Suppose that*

$$\frac{R_0 + \left[1 + \frac{1-\eta}{\psi_A}\right] Z_L}{2\left[1 + \frac{1-\eta}{\psi_P}\right]} < Z_M < Z_L \left(\frac{\psi_A + 1 - \eta}{\psi_P + 1 - \eta}\right) \frac{\psi_P}{\psi_A} \tag{89}$$

and that $W \geq \hat{W}$ *for some* $\hat{W} < \bar{W}$.[13] *Then* $W_A > 0$ *and* $W_P > 0$.

Proof of Lemma 6. Since $x_{AL} = \frac{2W_A}{P_L}$ and $x_P = \frac{W_P}{P_M}$, then $W_A > 0$ and $W_P > 0$ is equivalent to $x_{AL} > 0$ and $x_P > 0$. Using (71)-(72), this is equivalent to

$$\begin{cases} \left[1 + \frac{1-\eta}{\psi_A + (\lambda-1)(1-\eta)}\right] |Z_L| > \left[1 + \frac{1-\eta}{\psi_P + (\lambda-1)(1-\eta)}\right] |Z_M| \\ 2\left[1 + \frac{1-\eta}{\psi_P + (\lambda-1)(1-\eta)}\right] Z_M > R_0 + \left[1 + \frac{1-\eta}{\psi_A + (\lambda-1)(1-\eta)}\right] Z_L \end{cases} \tag{90}$$

Intuitively, Z_L are the trading gains captured by the active fund, and $Z_M < Z_L$ are the trading gains captured by the passive fund. If one fund's trading gains (relative to the costs of searching for that fund) are much larger than for the other, investors will not invest in the second fund.

(1) Let us start with the condition $x_{AL} > 0$, i.e, the first condition in (90). It is equivalent

[13] More precisely, the condition on $\hat{W}$ is that for $\hat{\lambda}$ corresponding to $W = \hat{W}$, we have $H_P(\hat{\lambda}) \equiv 2\left[1 + \frac{1}{\frac{\psi_P}{1-\eta} - 1 + \hat{\lambda}}\right] Z_M - R_0 - \left[1 + \frac{1}{\frac{\psi_A}{1-\eta} - 1 + \hat{\lambda}}\right] Z_L \geq 0$.

to

$$\left(\tfrac{\psi_A}{1-\eta}+\lambda\right)\left(\tfrac{\psi_P}{1-\eta}-1+\lambda\right) > \tfrac{Z_M}{Z_L}\left(\tfrac{\psi_P}{1-\eta}+\lambda\right)\left(\tfrac{\psi_A}{1-\eta}-1+\lambda\right) \Leftrightarrow$$
$$\Leftrightarrow \lambda^2\left(\tfrac{Z_L}{Z_M}-1\right)+\lambda\left(\tfrac{\psi_A}{1-\eta}+\tfrac{\psi_P}{1-\eta}-1\right)\left(\tfrac{Z_L}{Z_M}-1\right)+\left[\tfrac{\psi_A}{1-\eta}\left(\tfrac{\psi_P}{1-\eta}-1\right)\tfrac{Z_L}{Z_M}-\tfrac{\psi_P}{1-\eta}\left(\tfrac{\psi_A}{1-\eta}-1\right)\right] > 0$$
$$\Leftrightarrow \lambda^2+\lambda\left(\tfrac{\psi_A}{1-\eta}+\tfrac{\psi_P}{1-\eta}-1\right)+\frac{\left[\tfrac{\psi_A}{1-\eta}\left(\tfrac{\psi_P}{1-\eta}-1\right)\tfrac{Z_L}{Z_M}-\tfrac{\psi_P}{1-\eta}\left(\tfrac{\psi_A}{1-\eta}-1\right)\right]}{\tfrac{Z_L}{Z_M}-1} > 0 \Leftrightarrow \lambda^2+B\lambda+C>0,$$

where the second to last equivalence follows from $\frac{Z_L}{Z_M}-1>0$, and the last equivalence is simply a new notation. It can be shown that

$$B^2-4C\geq 0 \Leftrightarrow \frac{1}{4}\left[\frac{\psi_A}{1-\eta}-\frac{\psi_P}{1-\eta}\right]^2+\frac{1}{2}\frac{\psi_A}{1-\eta}-\frac{1}{2}\frac{\psi_P}{1-\eta}+\frac{1}{4}\geq -\frac{\frac{\psi_A}{1-\eta}-\frac{\psi_P}{1-\eta}}{\frac{Z_L}{Z_M}-1},$$

which always holds since $\psi_A>\psi_P$. Since $B^2-4C\geq 0$, a sufficient condition for $\lambda^2+B\lambda+C>0$ for all $\lambda\geq 1$ is that $\lambda_2<1$, where $\lambda_2=\frac{-B+\sqrt{B^2-4C}}{2}$, or equivalently, $\sqrt{B^2-4C}<2+B$. This requires (1) $B+2>0\Leftrightarrow \frac{\psi_A}{1-\eta}+\frac{\psi_P}{1-\eta}+1>0$, which always holds, and (2) $B^2-4C<B^2+4B+4\Leftrightarrow B+C+1>0$. Plugging in the expressions for B and C and simplifying, $B+C+1>0$ is equivalent to

$$\left(\frac{\psi_A+1-\eta}{\psi_P+1-\eta}\right)\frac{\psi_P}{\psi_A}\frac{Z_L}{Z_M}>1\Leftrightarrow\left(\frac{1+\frac{1-\eta}{\psi_A}}{1+\frac{1-\eta}{\psi_P}}\right)\frac{Z_L}{Z_M}>1. \tag{91}$$

(2) Next, consider the condition $x_P>0\Leftrightarrow$

$$H_P(\lambda)\equiv 2\left[1+\frac{1}{\frac{\psi_P}{1-\eta}-1+\lambda}\right]Z_M-R_0-\left[1+\frac{1}{\frac{\psi_A}{1-\eta}-1+\lambda}\right]Z_L>0.$$

If $H_P(1)>0$, then by continuity, there exists $\hat{\lambda}>1$ such that $H_P(\lambda)>0$ for all $\lambda\leq\hat{\lambda}$. Since λ is decreasing in W, $\lambda\leq\hat{\lambda}$ is equivalent to $W\geq\hat{W}$, i.e., investor wealth is not too limited for W below the cutoff $\bar{W}$. Hence, a sufficient condition for $x_P>0$ is $W\geq\hat{W}$ and $H_P(1)>0$, which is equivalent to

$$2\left[1+\frac{1}{\frac{\psi_P}{1-\eta}-1+\lambda}\right]Z_M-R_0-\left[1+\frac{1}{\frac{\psi_A}{1-\eta}-1+\lambda}\right]Z_L>0. \tag{92}$$

Combining (91) and (92) gives (89) and completes the proof. ■

Lemma 7 *Suppose $W<\frac{R_0-Z_L}{2}$. Then, given the equilibrium characterized by Proposition 1, liquidity investors are marginal for both stock L and stock H.*

Proof of Lemma 7. Note that $x_{AL}+x_P=\frac{2W_A}{P_L}+\frac{W_P}{P_M}$, where

$$\begin{aligned} P_L &= R_0-Z_L+c_A'^{-1}(f_Ax_{AL})+c_P'^{-1}(f_Px_P)\geq R_0-Z_L, \\ P_M &= R_0-Z_M+\frac{1}{2}c_A'^{-1}(f_Ax_{AL})+c_P'^{-1}(f_Px_P)\geq R_0-Z_M>R_0-Z_L. \end{aligned}$$

Hence,

$$x_{AL}+x_P\leq\frac{2W_A+W_P}{R_0-Z_L}\leq\frac{2(W_A+W_P)}{R_0-Z_L}=\frac{2W}{R_0-Z_L}.$$

It follows that the condition $W<\frac{R_0-Z_L}{2}$ ensures that $x_{AL}+x_P<1$, i.e., liquidity investors are marginal for stock L. This, in turn, implies $x_P<1$, i.e., liquidity investors are also marginal for stock H. ■

Lemma 8 (upper bound on λ) *In any equilibrium given by Proposition 1 or Lemma 1, it must be $\lambda\leq\lambda_{\max}$, where*

$$\lambda_{\max}=\begin{cases}\frac{R_0}{R_0-Z_L}-\psi_A, & \textit{if } W_A>0,\\ \frac{R_0}{R_0-Z_M}-\psi_P, & \textit{otherwise.}\end{cases} \tag{93}$$

Proof of Lemma 8.

First, suppose that $W_A>0$. Then,

$$\lambda=(1-f_A)\frac{R_L}{P_L}-\psi_A\leq\frac{R_L}{P_L}-\psi_A,$$

where

$$\frac{R_L}{P_L}=\frac{R_0+e_{AL}+e_P}{R_0-Z_L+e_{AL}+e_P},$$

and $\frac{R_0+e_{AL}+e_P}{R_0-Z_L+e_{AL}+e_P}\leq\frac{R_0}{R_0-Z_L}$ since $\frac{R_0+x}{R_0-Z_L+x}$ decreases in x. Hence, $\lambda\leq\lambda_{\max}$, as required.

Second, suppose that $W_A=0$ and $W_P>0$. Then,

$$\lambda=(1-f_P)\frac{R_M}{P_M}-\psi_P\leq\frac{R_M}{P_M}-\psi_P,$$

where

$$\frac{R_M}{P_M}=\frac{R_0+\frac{1}{2}e_{AL}+e_P}{R_0-Z_M+\frac{1}{2}e_{AL}+e_P},$$

and $\frac{R_0+\frac{1}{2}e_{AL}+e_P}{R_0-Z_M+\frac{1}{2}e_{AL}+e_P}\leq\frac{R_0}{R_0-Z_M}$ since $\frac{R_0+x}{R_0-Z_M+x}$ decreases in x. Hence, $\lambda\leq\lambda_{\max}$, as required. ■

Lemma 9 (decreasing the cost of monitoring when only one fund exists) *Consider* *the equilibrium of Lemma 1, in which only the passive (active) fund raises positive AUM. Then, the passive (active) fund manager's payoff always strictly decreases if c_P (c_A) decreases.*

Proof of Lemma 9.

The proof immediately follows from the proof of Proposition 4, because this statement has already been proved for the case $\lambda = 1$ in Proposition 4, and the proof applies to equilibria with only one fund as well. ■